Covariant and locally Lorentz-invariant varying speed of light theories

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We propose definitions for covariance and local Lorentz invariance applicable when the speed of light c is allowed to vary. They have the merit of retaining only those aspects of the usual definitions which are invariant under unit transformations, and which can therefore legitimately represent the outcome of an experiment. We then discuss some possibilities for invariant actions governing the dynamics of such theories. We consider first the classical action for matter fields and the effects of a changing c upon quantization. We discover a peculiar form of quantum particle creation due to a varying c. We then study actions governing the dynamics of gravitation and the speed of light. We find the free, empty-space, no-gravity solution, to be interpreted as the counterpart of Minkowksi space-time, and highlight its similarities with Fock-Lorentz space-time. We also find flat-space string-type solutions, in which near the string core c is much higher. We label them fast-tracks and compare them with gravitational wormholes. We finally discuss general features of cosmological and black hole solutions, and digress on the meaning of singularities in these theories.

I. INTRODUCTION

The varying speed of light (VSL) theory provides an elegant solution to the cosmological problems - the horizon, flatness, and Lambda problems of Big-Bang cosmology. The theory has appeared in several guises ([1–18])), but in the formulation proposed by Albrecht and Magueijo [2] (see also [3–7,13,14]) one finds the most direct mechanism for converting the Einstein deSitter model into a cosmological attractor. Unfortunately the foundations of such a theory are far from solid. Covariance and local Lorentz invariance are explicitly broken, and are not replaced by similar far-reaching principles. The difficulty in applying the theory to situations other than cosmology (eg. black holes) stems directly from this deficiency.

This paper is an attempt to remedy this shortcoming. This may be achieved in various different ways, some of which inevitably rather radical. We note that nothing prevents the construction of a theory satisfying the principle of relativity, while still allowing for space-time variations in c. Such a theory would in general not be Lorentz invariant, but it could still be relativistic. Indeed, Lorentz invariance follows from two independent postulates: the principle of relativity and the principle of constancy of the speed of light. Dropping the latter while keeping the former leads to a new invariance, known as Fock-Lorentz symmetry [19–21]. This invariance does not distinguish between inertial frames (and therefore satisfies the principle of relativity) but it allows for a varying c; indeed it allows for a non-invariant c.

A possible approach is therefore to set up a theory of gravitation based upon a gauged Fock-Lorentz symmetry. However we note that such an enterprise accommodates more than is required by VSL theories: it allows the speed of light at a given point to depend on the observer's speed. Also the speed of light in the Fock-Lorentz space is anisotropic. Clearly, certain aspects of the second postulate of Einstein's relativity theory may be kept in the simplest VSL theories, namely that the speed of light at a given point be independent of its color, direction, or the speeds of either emitter or observer.

In this paper we shall be as conservative as possible and preserve all aspects of the second postulate of special relativity consistent with allowing space-time variations of c. In Section II we show that such a reformulation gleans from the second postulate of relativity all that is operationally meaningful, in the sense that the aspects of the second postulate which we preserve are exactly those which can be the outcome of experiment (such as the Michelson-Morley experiment). The constancy of c in space-time, on the other hand, amounts to nothing more than a definition of a system of units. In Section II and III we show that such a theory is *locally* Lorentz invariant and generally covariant, subject to a minimal generalization of these concepts. In Section IV we summarise the overall structure of such theories, and the basic reasons for adopting it.

We then discuss Lagrangians governing the dynamics of these theories. The main practical drawback of explicit lack of covariance is that it makes an action principle formulation rather awkward (see [4,18]). The VSL theories proposed in this paper, on the contrary, are easily amenable to an action principle formulation. However we shall try to borrow some features from earlier models, such as lack of energy conservation.

In Section V we first consider the matter action. We show how it is always possible to define the matter Lagrangian so that c does not appear explicitly. Such a principle fixes a large number of scaling laws for other "constants" as a function of c. It also leads to simpler dynamical equations for c.

Two constants are left undetermined by these considerations: Planck's constant $\hbar$ and Boltzmann's constant k_B. These cannot be determined by classical dynamics, and scaling laws $\hbar(c)$ and $k_B(c)$ should be postulated. In Section VI we consider the implications of various $\hbar(c)$

for quantization. We identify situations in which a varying speed of light leads to quantum particle creation.

Then in Section VII we consider Lagrangians for gravitational dynamics (we include the dynamics of c into this discussion - as the field c can be seen as an extra gravitational field). We identify the actions which lead to nothing but a change of units in a standard Brans-Dicke theory; all other actions are intrinsically different theories.

The rest of the paper is devoted to the simplest applications of these theories. In Section IX we discuss empty space solutions. We find a variation in c and a global space-time which is very similar to those found in Fock-Lorentz space. We also show how Fock-Lorentz space is nothing but a change of units applied to Minkowski space-time. However $t = \infty$ is brought to a finite time in the varying c representation. We show that the space is actually extendable beyond this finite time; into what in the fixed c representation would be a trans-eternal region.

Another flat-space solution to our theory is a soliton string, close to which the speed of light is much larger. We label it a fast-track. A spaceship moving along a fast track could move at non-relativistic speeds, without a twin paradox effect, and still cover enormous intergalactic distances. These solutions are not dissimilar to gravitational wormholes; and indeed they are mapped into wormhole-like structures in fixed c units.

We finally discuss general features of cosmological solutions and black holes in these theories. These will be developed further in two publications currently under preparation [22,23]. Concerning black holes the main novelty is that for some regions of the couplings the speed of light may go to zero at the horizon. This effectively prevents any observer from entering the horizon, and its interior should therefore be excised from the manifold. We relabel this boundary an "edge", and comment on the implication of this effect for a generalized cosmic censorship principle.

II. GENERALIZED LORENTZ INVARIANCE

From an operational point of view all laws of physics should be invariant under global and local changes of units [24,25,2]. Indeed measurements are always ratios to standard units, and therefore represent essentially dimensionless quantities. Physics should therefore be dimensionless or unit-invariant. However, this far-reaching principle is rarely incorporated into theoretical constructions, because a concrete choice of units usually simplifies the statement of laws. While this practical consideration should be recognized, it is important to realize that some theoretical constructions are tautological, and amount to nothing more than the specification of a system of units.

An example is the second postulate of special relativity: the constancy of c. Clearly the postulate is invariant under unit transformations when it states that light of different colors travels at the same speed - as it makes a statement about the ratio of two speeds at a given point, which is a dimensionless quantity. The postulate is also unit-independent when it incorporates the result of the Michelson-Morley experiment: light emitted by sources moving at different speeds travels at the same speed. Again it makes use of ratios of speeds: the ratio of the sources' speeds, and the ratios of the different light rays' speeds. However the postulate looses its meaning when it refers to light speed at different points, or even to the speed of light moving in different directions at a given point.

Hence Lorentz invariance in its usual definition is not a unit-independent concept, and indeed relativity is not a unit-independent construction. For instance relativity is not conformally invariant (a conformal transformation being just a particular type of unit transformation). A unit-independent definition of Lorentz invariance may be inferred by taking a Lorentz invariant theory and subjecting it to the most general unit transformation. The resulting theory retains the unit-invariant aspects of the second postulate, and clearly c may now be anisotropic and vary in space-time. Under such circumstances what is the structure which represents Lorentz invariance?

For simplicity we specialize to changes of units which only affect the local value of the modulus of c. We redefine units of time and space in all inertial systems

$$d\hat{t} = dt\epsilon^{\alpha}$$
$$d\hat{x}^i = dx^i\epsilon^{\beta} \qquad (1)$$

where ϵ can be any function, and the metric (in this case the Minkowski metric) is left unchanged. If $\alpha = \beta$ we have an active conformal transformation (for a passive conformal transformation dx and dt are left unchanged, and the metric is multiplied by $\epsilon^{2\alpha}$). If $\alpha \neq \beta$, a Lorentz invariant theory is replaced by a theory in which c remains isotropic, color independent, and independent of the speeds of observer and emitter; but it varies like $\hat{c} \propto \epsilon^{\beta-\alpha}$. A general unit transformation may be decomposed into a conformal transformation plus a VSL transformation with $\beta = 0$.

It is immediately obvious that local Lorentz transformations in the new units are preserved:

$$d\hat{t}' = \gamma\left(d\hat{t} - \frac{\hat{v}}{\hat{c}^2}d\hat{x}\right)$$
$$d\hat{x}' = \gamma\left(d\hat{x} - \hat{v}d\hat{t}\right) \qquad (2)$$

with

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{\hat{v}}{\hat{c}}\right)^2}} \qquad (3)$$

Hence the standard definition remains unmodified, if one employs the local value of c in the transformation.

A novelty arises because changes of space-time units do not generally produce new coodinate patches because

2

(1) needs not be holonomic: one may have eg. $d^2\hat{t} \neq 0$. Hence there would not be a global $\hat{t}$ time coordinate: the new "coordinate elements" would not be differentials of any coordinates. Even if in one frame the transformation (1) were holonomic, in a boosted frame it would not be. Some oddities pertaining to the new units follow. Partial derivatives generally do not commute. The change in the "coordinate time" between two points may depend upon the path taken to link the two points.

We have thus identified the structure of a VSL Lorentz invariant theory. The theory is locally Lorentz invariant in the usual way, using in local transformations the value of c at that point. However local measurements of time and space are not closed forms, and therefore cannot be made into coordinates. Integrating factors can always be found, so that $d\hat{t}/\epsilon^\alpha$ and $d\hat{x}/\epsilon^\beta$ are closed forms, and $\hat{c} = \epsilon^{\beta-\alpha}$.

Although a time coordinate does not generally exist, in many important cases it may be defined. If $\partial_\mu c \partial^\mu c < 0$ then local coordinates exist so that c only changes in time. We shall call this the homogeneous frame. Then $d^2\hat{t} = 0$, and a $\hat{t}$ coordinate can be defined. Hence if we insist upon using a time coordinate we necessarily pick up a preferred reference frame - thereby violating the principle of relativity. This situation will be true in cosmology (where the preferred frame is the cosmological frame) but not in the context of static solutions, such as black hole solutions. Also a time coordinate may always be defined along a line. In particular for a geodesic, the amount of proper time is always well defined, although the proper time between two points depends on the trajectory (a situation already true in general relativity).

III. GENERALIZED COVARIANCE

In order to construct a theory of gravitation we need to discuss general covariance. Covariance is the requirement of invariance under the choice of coordinate chart. This may be trivially adapted to VSL if we only use charts employing an "x^0" coordinate, with dimensions of length rather than time. Then c appears nowhere in the usual definitions of differential geometry, which may therefore still be used. The laws for the transformation of tensors are the same as usual. The metric is dimensionless in all components and does not explicitly depend on c; it transforms like a rank 2 tensor. The usual Cristoffell connection may be defined from the metric by means of the standard formula, without any extra terms in the gradients of c (which only appear if we try to revert to a time type of coordinate). A curvature tensor may still be defined in the usual way, and a Ricci tensor and scalar derived from it. The volume measure does not contain c. As we will see, many novelties introduced by a varying c only emerge when we try to connect the x^0 coordinate with time.

Whenever applying a unit transformation (1) to a co-variant theory, the above remarks apply only to the VSL part of the transformation (that is the component with $\beta = 0$). For the conformal part of the transformation, with $\epsilon^\alpha = \epsilon^\beta = \Omega$, the structures of differential geometry transform in the usual way [26]. For instance the Ricci scalar transforms as:

$$\hat{R} = \frac{R}{\Omega^2} - 6\frac{\Box\Omega}{\Omega^3} \qquad (4)$$

for active conformal transformations.

It is not altogether surprising that covariance may be redefined so easily for a theory with such different foundations. It has been pointed out that covariance is an empty requirement (see [27,28]). Not only does covariance not imply local Lorentz invariance, but also *any* theory can be made covariant. An example of a covariant formulation of Newtonian gravity is given in [28]. In this theory the tangent space is not a portion of Minkowski space, rather a portion of Galilean space.

IV. AN OVERVIEW OF THE UNDERLYING STRUCTURE

What structure represents covariance and local Lorentz invariance when c is allowed to vary? We found that it is a unit-invariant redefinition of these concepts, which indeed does not differ much from the usual definitions if we phrase them suitably. Local Lorentz transformations are the same as usual, using the local value of c. Covariance and the usual constructions of differential geometry remain unchanged as long as a x^0 coordinate is used, or more generally if all coordinates used have the same dimensions.

What is new, then? The novelty is that locally made time and space measurements produce a set of infinitesimals which are generally not closed forms. Therefore space-time measurements cannot be made into local coordinate patches. This leads to the following modification of the structure of relativity. The underlying structure of general relativity is a manifold, combined with its tangent bundle (where physics actually happens). If c varies the underlying structure is a fibre bundle. The base manifold has the same structure as usual, but the fibres in which local measurements happen are not the tangent bundle. The fibers are vector spaces obtained by means of a non-holonomic transformation over the tangent bundle.

It may seem rather contorted to adopt the above structure when we know that a unit transformation would transform it into standard covariance and local Lorentz invariance. However such a structure has the merit that it only incorporates those elements of the original structure which are unit-independent, and can therefore be the outcome of experiment. Moreover such structure allows for a varying speed of light within a covariant framework, which is precluded by the standard framework. What one may gain from such extra freedom is a simplified description of any given physical situation, when all fine

3

structure coupling constants are allowed to vary, in what looks like a contrived fashion, if we use units such that c is constant.

We wish to propose a theory which permits space-time variations in all coupling constants; more specifically in generalized fine structure constants $\alpha_i = g_i^2/(\hbar c)$ - where g_i are the various charges corresponding to all interactions apart from gravitation. This purpose draws inspiration from the findings of [29]. However we restrict such variations so that the ratios between the α_i remain constant. This suggests that attributing the variations in the α_i to changes in c or $\hbar$ might lead to a simpler picture. In suitable units we could regard our theory as a "generalized " Bekenstein changing e theory, but in this system of units the picture is rather contrived.

We will see, in Section VIII, that a natural dynamics will emerge in this theory which becomes unnecessarily complicated when the theory is reformulated in fixed c units. Whatever the system of units chosen the general theory we will consider is not a dilaton theory. Some important geometrical aspects (such as inaccessible regions of space-time to be studied in Section IX) are missed altogether in the fixed c system.

V. MATTER FIELDS SUBJECTED TO VSL

Before embarking on an investigation of the dynamics of c and of gravitation, we first undertake a careful examination of the effects of a varying c upon the matter fields. The key point here is that it is always possible to write the matter Lagrangian so that is does not depend explicitly on c. We may break this rule, if we wish to, but this is not necessary. This remark is highly non-trivial, and relies heavily on using an x^0 coordinate (as opposed to time). The introduction of a time coordinate would not only introduce non-covariant elements in expressions like $\partial_\mu \phi = (\partial_t \phi/c, \partial_i \phi)$, but would also force the matter Lagrangian to depend explicitly on c, via kinetic terms.

Using an x^0 coordinate the situation is rather different. For instance, for a massless scalar field with no interactions we have:

$$\mathcal{L}_m = -\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) \tag{5}$$

which does not depend on c. Similarly for a spin 1/2 free massless field we have

$$\mathcal{L}_m = i\overline{\chi}\gamma^\mu \nabla_\mu \chi \tag{6}$$

The above expressions, in particular the latter, are sometimes multiplied by $\hbar c$ (see for example Mandl and Shaw [30]). If c and $\hbar$ are constant this operation has no effects, other than modifying the dimensions of the fields. However in a minimal VSL theory such an operation should be banned. All dynamical fields should be defined with dimensions such that the kinetic terms have no explicit

dependence on either c or $\hbar$. This forces all matter fields to have dimensions of $\sqrt{E/L}$.

The only chance for $\mathcal{L}_m$ to depend upon c therefore comes from mass and interaction terms. These may always be defined so that no explicit dependence on c is present. By dimensional analysis this requirement fully defines how masses, charges, and coupling constants scale with c, provided we know how $\hbar$ scales with c. This issue will be discussed further in the next Section.

Let us first consider mass terms. For a scalar field we have:

$$\mathcal{L}_m = -\frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi + \frac{1}{\lambda_\phi^2}\phi^2\right) \tag{7}$$

Hence in a minimal VSL theory the Compton wavelength λ_ϕ of the particle should not depend on c. For a massive spin 1/2 particle we have

$$\mathcal{L}_m = i\overline{\chi}\gamma^\mu \nabla_\mu \chi - \frac{1}{\lambda_\chi}\overline{\chi}\chi \tag{8}$$

with a similar requirement. More generally we find that c does not appear in mass terms if all particles' masses are proportional to $\hbar/c$ (or their rest energies proportional to $\hbar c$).

If we now consider fields coupled to electromagnetism we find that the electric charge e should scale like $\hbar c$, if explicit dependence on c is to be avoided. Consider for instance a $U(1)$ gauged complex scalar field. Its action may be written as

$$\mathcal{L}_m = -(D^\mu \phi)^\star D_\mu \phi - \frac{|\phi|^2}{\lambda_\phi^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{9}$$

where the $U(1)$ covariant derivative is

$$D_\mu = \partial_\mu + i\frac{e}{\hbar c}A_\mu \tag{10}$$

and the electromagnetic tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{11}$$

Hence e should be proportional to $\hbar c$. The same holds true for fields of any spin coupled to electromagnetism, since e only appears in the definition of the covariant derivative. Notice that the constancy of $e/(\hbar c)$ is also required for the gauge-invariant field strength tensor not to receive any corrections. Gauge transformations should take the form:

$$\delta A_\mu = -\frac{\hbar c}{e}\partial_\mu f \tag{12}$$

for $\delta\phi = if\phi$, where f is any function. This is necessary so that $D_\mu \phi$ transforms covariantly: $\delta(D_\mu \phi) = if D_\mu \phi$. But then the gauge invariant field strength tensor must be defined as

$$F_{\mu\nu} = \frac{\hbar c}{e}\left(\partial_\mu\left(\frac{e}{\hbar c}A_\nu\right) - \partial_\nu\left(\frac{e}{\hbar c}A_\mu\right)\right) \tag{13}$$

4

and indeed this receives extra terms if $e/(\hbar c)$ is not constant.

Inspection of the electroweak and strong interaction Lagrangians reveals that their coupling charges g should also scale like $\hbar c$. This is indeed a general feature for any interaction, and follows from dimensional analysis. It is always the combination $g/(\hbar c)$ that appears in covariant derivatives and, in non-Abelian theories, in the gauge field strength tensor.

Next we discuss two important cases to be used later in this paper: a field undergoing spontaneous symmetry breaking, and a matter cosmological constant. Consider a U(1) gauge symmetric complex scalar field as above, but with a potential

$$V(\phi) = \frac{1}{\lambda_\phi^2}|\phi|^2 - \frac{1}{2\lambda_\phi^2\phi_0^2}|\phi|^4 \qquad (14)$$

Then the Compton wavelength λ_ϕ and ϕ_0 should both be independent of c. If the quartic term is ignored then the vacuum is at $\phi = 0$, so that we have a massive complex scalar field (with Compton wavelength λ_ϕ), and a massless gauge boson. The charge is $e \propto \hbar c$. If we consider the quartic term, as is well known, we have spontaneous symmetry breaking. The vacuum is now at $|\phi| = \phi_0$. Expanding around the vacuum we find a real scalar field with Compton wavelength λ_ϕ, and a massive gauge boson with Compton wavelength

$$\frac{1}{\lambda_A} = \frac{e}{\hbar c}\phi_0 \qquad (15)$$

which therefore is independent of c. Hence the rest energies of all massive particles, regardless of the origin of their mass, scale like $\hbar c$. Due to spontaneous symmetry breaking the vacuum energy decreases by

$$\Delta V = -\frac{\phi_0^2}{2\lambda^2} \qquad (16)$$

and so this process gives rise to a negative vacuum energy, if the original vacuum energy is zero. We shall label it by $\Lambda_m = \Delta V$, and call it the matter cosmological constant. It adds a term to the matter Lagrangian

$$\mathcal{L}_m = -\Lambda_m \qquad (17)$$

Under minimal coupling Λ_m does not depend on c. However we could also allow ϕ_0, and therefore Λ_m, to depend on c (as we shall do in [22]).

Finally we consider an example of a classical Lagrangian, that of a charged particle in a field:

$$\mathcal{L}_m(x^\gamma) =$$
$$\int d\lambda \left[-\frac{E_0}{2}\frac{dy^\mu}{d\lambda}\frac{dy_\mu}{d\lambda} + eA_\mu\frac{dy^\mu}{d\lambda} \right] \frac{\delta^{(4)}(x^\gamma - y^\gamma)}{\sqrt{-g}} \qquad (18)$$

Here the affine parameter is $d\lambda = cd\tau$, where τ is proper time. Note that any of the variations sometimes employed in the literature, eg. using the square root of $-u^2$

(with $u = dx/d\lambda$), should not be used. This is because, as we shall see, u^2 needs not remain constant. Minimal coupling therefore requires that the particle's rest energy $(E_0 = m_0c^2)$ and charge e be independent of c.

In non-minimal theories we may consider a direct dependence on c in the matter Lagrangian. This is far from new: for instance Bekenstein's theory [31] allows for a direct coupling between a varying e and all forms of matter coupled to electromagnetism.

A. A worked out example

Consider a massive scalar field ϕ in flat space-time (metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ if we use a x^0 coordinate) with a variation in c such that

$$c = \frac{c_0}{1 + \frac{c_0 t}{R}} \qquad (19)$$

in suitable coordinates (so that c does not vary in space). We have defined c_0 as the speed of light at time $t = 0$. At time $t = -R/c_0$ the speed of light goes to infinity. As time progresses the speed of light decays to zero, as $t \to \infty$. We shall see that this is indeed the solution corresponding to flat space-time. Then ϕ satisfies:

$$\ddot{\phi} - \nabla^2\phi + \frac{1}{\lambda_\phi^2}\phi = 0 \qquad (20)$$

which may be solved with Fourier series, with amplitudes subject to

$$\ddot{\phi}_k + \left(k^2 + \frac{1}{\lambda_\phi^2} \right)\phi = 0 \qquad (21)$$

The solution is

$$\phi_k = \phi_0(k)e^{i(\pm k^0 x^0 + \mathbf{k}\cdot\mathbf{x})} \qquad (22)$$

with a dispersion relation

$$(k^0)^2 = k^2 + \frac{1}{\lambda_\phi^2} \qquad (23)$$

As expected there is nothing new if we use a x^0 coordinate.

If we insist on using a time coordinate we find that we can only do so in one inertial frame, the one in which the speed of light is homogeneous. By requesting to use a time coordinate, and make contact with physics, we therefore select a preferred reference frame. In this frame:

$$x^0 = \int c\,dt = R\log\left(1 + \frac{c_0 t}{R}\right) \qquad (24)$$

and so we find that around a given time $t = t_0$ we have the Taylor expansion:

$$k^0 x^0 = k^0 c(t_0)(t - t_0) + k^0 R \log \left(1 + \frac{c_0 t_0}{R}\right) \qquad (25)$$

We find that the local frequency changes proportionally to c:

$$\omega(t) = k^0 c(t) = \frac{k^0 c_0}{1 + \frac{c_0 t}{R}} \qquad (26)$$

In addition there is a phase shift with value

$$\Phi_0 = k^0 R \log \left(1 + \frac{c_0 t_0}{R}\right) \qquad (27)$$

As we approach the initial singularity the wave suffers infinite blueshift. As time flows it redshifts progressively. The similarities between this effect and the cosmological redshift have been pointed out in [20]. However the effect presented here is not due to gravity (expansion) but is due purely to the varying speed of light.

Naturally the above identification of a local frequency is only valid if $\omega \gg |\dot{c}/c|$. This amounts to requiring:

$$k^0 \gg \frac{1}{R\left(1 + \frac{c_0 t}{R}\right)} \qquad (28)$$

Hence any plane-wave approximation breaks down near the initial singularity; an interesting result.

VI. QUANTIZATION

Unfortunately the requirement that c does not appear explicitly in $\mathcal{L}_m$ does not fix the scaling with c of all "constants": Planck's and Boltzmann's constants, $\hbar$ and k_B, are left unfixed. Furthermore these two "constants" cannot be fixed by the classical dynamics, i.e. by adding to the action dynamical terms in two scalar fields $\hbar$ and k_B (as we shall do with c). Instead these have to be provided as a function of c, by means of scaling laws $\hbar(c)$ and $k_B(c)$. These scaling laws should be regarded as postulates of the theory.

Here we explore the implications of various $\hbar(c)$ laws. Let us consider first the simple case of a non-relativistic linear harmonic oscillator. Its Lagrangian is given by:

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \qquad (29)$$

and we assume that m and ω are independent of c, and therefore constant. Its Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{m \omega^2 x^2}{2} \qquad (30)$$

and is time-independent. We postulate that quantization produces an expression of the form

$$\hat{H} = \hbar \omega (\hat{N} + 1/2) \qquad (31)$$

where $\hat{N}$ is the particle number operator, ie: the classical energy of the oscillator is in quanta of energy $\hbar\omega$. Hence

$$\frac{d}{dt} \hbar(\hat{N} + 1/2) = 0 \qquad (32)$$

This implies that should $\hbar$ drop, the particle number would increase, which is hardly surprising. Indeed the amplitude A and frequency ω of the classical oscillations remain constant, and therefore so does their total energy $E = m \omega^2 A^2/2$. However the quantum particles contained in the oscillator have energies $\hbar\omega$ which vary like $\hbar$. To reconcile these two facts the number of particles has to vary, proportionally to $1/\hbar$. Such a phenomenon has a clear experimental meaning, since the number of particles does not depend on the units being used.

Furthermore if the oscillator is initially in the vacuum state, a drop in $\hbar$ suppresses the zero-point energy. Particles should therefore be produced so that $\hat{H}$ remains constant. We have both particle multiplication and particle production (a phenomenon noticed before in VSL theories by [15]).

Since creation and anihilation operators satisfy a time-independent algebra $[a, a^\dagger] = 1$, the best way to express the variability of $\hat{N}$ is by means of a Bogolubov-type transformation. A short calculation shows that:

$$a(t') = \alpha a(t) + \beta^\star a^\dagger(t) \qquad (33)$$

with

$$|\alpha|^2 = \frac{\hbar(t) + \hbar(t')}{\hbar(t')} \qquad (34)$$

$$|\beta|^2 = \frac{\hbar(t) - \hbar(t')}{\hbar(t')} \qquad (35)$$

enforces that the expectation values of $\hat{N}(t) = a^\dagger(t)a(t)$ satisfy (32).

This discussion generalizes to relativistic quantum field theory, with $\hbar c$ replacing $\hbar$. Now we should have:

$$\hat{H} = \sum_{\mathbf{k}} \hbar \omega (\hat{N} + 1/2) \qquad (36)$$

with $\omega = k^0 c \propto c$. Hence now

$$\frac{d}{dx^0} \hbar c (\hat{N} + 1/2) = 0 \qquad (37)$$

The time dependence in $\hat{N}$ can now be expressed in the form of a Bogolubov transformation

$$a(k^0, x^{\overline{\mu}}) = \alpha a(k^0, x^\mu) + \beta^\star a^\dagger(k^0, x^\mu) \qquad (38)$$

with

$$|\alpha|^2 = \frac{\hbar(x^\mu)c(x^\mu) + \hbar(x^{\overline{\mu}})c(x^{\overline{\mu}})}{\hbar(x^\mu)c(x^\mu)} \qquad (39)$$

$$|\beta|^2 = \frac{\hbar(x^\mu)c(x^\mu) - \hbar(x^{\overline{\mu}})c(x^{\overline{\mu}})}{\hbar(x^\mu)c(x^\mu)} \qquad (40)$$

Recalling that $g_i/(\hbar c)$ is a constant, we have particle production at a rate proportional to $1/\alpha_i$, where i labels the various interactions. We shall parameterize $\hbar(c)$ by means of an exponent q such that

$$\alpha_i \propto g_i \propto \hbar c \propto c^q \qquad (41)$$

VII. GRAVITATIONAL DYNAMICS

We now set up some possibilities for actions governing the evolution of the metric and speed of light. Only a small class of these actions may be transformed into a dilaton action, by means of a unit transformation. In Appendix we describe a somewhat orthogonal approach.

We shall take as our starting point the action of General Relativity:

$$S = \int d^4x \sqrt{-g}\left(R - 2\Lambda + \frac{16\pi G}{c_0^4}\mathcal{L}_m\right) \qquad (42)$$

where R is the Ricci scalar, and Λ is the geometrical cosmological constant (as defined in [33,2]) and $\mathcal{L}_m$ is the Lagrangian of all the matter fields (including the above mentioned matter cosmological constant).

A changing G theory was proposed by Brans and Dicke [32], and we shall work in analogy to this generalization of General Relativity in what follows, albeit with a couple of crucial differences. The idea in this paper (in [32]) is to replace c (G) by a field, wherever it appears in (42). In addition one should add a term to the Lagrangian describing the dynamics of c (G). An ambiguity appears because (42) may be divided by any power of c (G), before the replacement is performed. Brans and Dicke avoided commenting on this ambiguity, and cunningly performed the necessary division by G which led to a theory with energy conservation. We shall not be hampered by this restriction; indeed we expect violations of energy conservation in VSL. Hence we consider actions in which the replacement is made after the most general division by c is made. In the simplest case we define a scalar field

$$\psi = \log\left(\frac{c}{c_0}\right) \qquad (43)$$

so that $c = c_0 e^\psi$, and take

$$S = \int d^4x \sqrt{-g}(e^{a\psi}(R - 2\Lambda + \mathcal{L}_\psi) + \frac{16\pi G}{c_0^4}e^{b\psi}\mathcal{L}_m) \qquad (44)$$

The simplest dynamics for ψ derives from:

$$\mathcal{L}_\psi = -\kappa(\psi)\nabla_\mu\psi\nabla^\mu\psi \qquad (45)$$

where $\kappa(\psi)$ is a dimensionless coupling function (to be taken as a constant in most of what follows). We shall impose $a - b = 4$, although this is not necessary.

Notice that $a = 4$, $b = 0$, is nothing but a unit transformation applied to Brans-Dicke theory, with

$$\phi_{bd} = \frac{e^{4\psi}}{G} \qquad (46)$$

$$\kappa(\psi) = 16\omega_{bd}(\phi_{bd}) \qquad (47)$$

This shall be proved in Section VIII, where we identify the full set of cases which are a mere unit transformation applied to existing theories. Among the theories which are truly new, $a = 0$, $b = -4$ is particularly simple and we shall call it minimal VSL.

We can trivially generalize this construction, by complicating the dynamics encoded in $\mathcal{L}_\psi$, for instance by adding a potential $V(\psi)$ to it. We can also take for ψ a complex, vector, or spinor field, with the speed of light deriving from a scalar associated with ψ (eg. $\overline{\psi}\psi$ for a spinor field). A nice example (developed further in Section X) is a theory in which ψ is a complex field, with

$$c = c_0 e^{-|\psi|^2} \qquad (48)$$

and with a Mexican hat potential added to $\mathcal{L}_\psi$.

Another important novelty of our theory, not included in Brans-Dicke theory (but noted by [33]), is that we allow Λ and Λ_m to depend on c. It seems fair to allow Λ, like $\hbar$ or k_B, to depend on c. After all Λ is a much less fundamental constant. On the contrary if Λ_m depends on c, then so does the vacuum expectation value ϕ_0, and so we have gone beyond minimal matter coupling. In what follows we shall absorb Λ_m into a total geometrical Lambda

$$\overline{\Lambda} = \Lambda + \frac{8\pi G}{c^4}\Lambda_m \qquad (49)$$

In our applications to cosmology [22] we shall assume that

$$\Lambda \propto (c/c_0)^n = e^{n\psi} \qquad (50)$$

and

$$\Lambda_m \propto (c/c_0)^m = e^{m\psi} \qquad (51)$$

We will see that allowing Λ to depend on c leads to interesting cosmological scenarios [22]. In such theories it is the presence of a Lambda problem that drives changes in the speed of light. These in turn solve the cosmological constant and other problems of Big Bang cosmology. In effect Lambda acts as a potential driving ψ.

A. Gravitational field equations

The field equations in this theory may now be derived by varying the action. Variation with respect to the metric leads to gravitational equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \kappa\left(\nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}\nabla_\delta\psi\nabla^\delta\psi\right)$$
$$+ e^{-a\psi}(\nabla_\mu\nabla_\nu e^{a\psi} - g_{\mu\nu}\Box e^{a\psi}) \tag{52}$$

where the matter stress energy tensor is defined as usual:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \tag{53}$$

These equations are particularly simple for minimal VSL ($a = 0$ and $b = -4$).

Variation with respect to ψ leads to

$$\Box\psi + a\nabla_\mu\psi\nabla^\mu\psi$$
$$= \frac{8\pi G}{c^4(2\kappa + 3a^2)}(aT - 2a\rho_\Lambda - 2b\mathcal{L}_m) + \frac{1}{\kappa}\frac{d\overline{\Lambda}}{d\psi} \tag{54}$$

Again minimal VSL is particularly simple:

$$\Box\psi = \frac{32\pi G}{c^4\kappa}\left(\mathcal{L}_m - \left(1 - \frac{m}{4}\right)\Lambda_m\right) + \frac{1}{\kappa}n\Lambda \tag{55}$$

As announced above, in general either a matter or a geometrical Lambda drive changes in c. The total matter Lagrangian $\mathcal{L}_m$ also drives changes in c, if $b \neq 0$. Ambiguities in writing $\mathcal{L}_m$ (total divergences) are therefore relevant for c, as indeed for the matter field equations under VSL (see below).

B. Impact upon matter field equations

Bianchi identities applied to (52) and (54) imply

$$\nabla_\mu(T_\nu^\mu e^{b\psi}) = be^{b\psi}\mathcal{L}_m\nabla_\nu\psi \tag{56}$$

or equivalently:

$$\nabla_\mu T_\nu^\mu = -b(T_\mu^\nu - \delta_\mu^\nu\mathcal{L}_m)\nabla_\nu\psi \tag{57}$$

Therefore we only have energy conservation if $a = 4$, $b = 0$. In all other cases a varying c creates or destroys energy; indeed beyond the naive expectation (the term in $\mathcal{L}_m$ is far from expected). This fact merely reflects the interaction between the matter fields and the gravitational field ψ, present due to the coupling $e^{b\psi}\mathcal{L}_m$. This interaction affects the field equations for matter, beyond what was descrobed in Section V (which is only strictly correct if $b = 0$). Indeed taking the variation with respect to matter fields, in every situation where it is usual to neglect a full divergence, a new term in $\partial^\mu\psi$ now appears. For instance scalar fields satisfy a modified Klein-Gordon equation:

$$\left(\Box - \frac{1}{\lambda_\phi^2}\right)\phi = -b\nabla_\mu\phi\nabla^\mu\psi \tag{58}$$

with gradients of ψ driving the field ϕ and therefore changing its energy balance. All field equations will be similarly affected, with a net result that energy conservation is violated according to (56).

To give a concrete example, the plane wave solution studied in Section V A is now subject to:

$$\ddot{\phi}_k + \left(k^2 + \frac{1}{\lambda_\phi^2}\right)\phi = -b\frac{\dot{\phi}_k}{R} \tag{59}$$

A solution is

$$\phi_k = [\phi_0(k)e^{-\frac{bx^0}{R}}]e^{i(\pm k^0 x^0 + \mathbf{k}\cdot\mathbf{x})} \tag{60}$$

subject to the same dispersion relation. Hence, in addition to the effects studied in Section V A, the amplitude of the plane waves is now proportional to c^b. If $R > 0$, and $b > 0$, we not only have a "redshift effect" (affecting the energy of the field quanta), but the classical energy of the field also dissipates.

Finally note that we may also take on board terms which are usually neglected in minimal theories because they are full divergences. If $b \neq 0$ these terms affect the matter field equations; indeed they drive changes in c. For instance one could consider electromagnetism based on

$$\mathcal{L}_m = -\frac{1}{4}(F_{\mu\nu}F^{\mu\nu} + \zeta F_{\mu\nu}\tilde{F}^{\mu\nu}) \tag{61}$$

where $\tilde{F}$ is the dual of F and ζ is a constant. The second term is usually irrelevant, because it is a full divergence. However we now have Maxwell's equations:

$$\nabla_\mu F^{\mu\nu} + b(F^{\mu\nu} + \zeta\tilde{F}^{\mu\nu})\partial_\mu\psi = j^\nu \tag{62}$$

where j^ν is the electric current.

C. Effect upon classical particles

These processes are also reflected in the equations of motion for a point particle. From (18), with $e = 0$, we can derive the stress energy tensor:

$$T^{\mu\nu}(x^\delta) = mc^2\int d\lambda\frac{dy^\mu}{d\lambda}\frac{dy^\nu}{d\lambda}\frac{\delta^{(4)}(x^\delta - y^\delta(\lambda))}{\sqrt{-g}} \tag{63}$$

where we have assumed that mc^2 is a constant (so that the matter Lagrangian does not depend on c). From (56) one gets:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\delta}\frac{dx^\nu}{d\lambda}\frac{dx^\delta}{d\lambda} = -b\left(\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{1}{2}\frac{dx^\alpha}{d\lambda}\frac{dx_\alpha}{d\lambda}g^{\mu\nu}\right)\psi_{,\nu} \tag{64}$$

where we recall $d\lambda = cd\tau$. Alternatively we may integrate the volume integral in (18) to obtain action:

$$S = -\frac{E_0}{2}\int d\lambda e^{b\psi}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{65}$$

Direct variation of this action is equivalent to equation (64) and may be more practical. An immediate first integral of this action is:

$$u^2 = u_0^2 (c/c_0)^{-b} \qquad (66)$$

with $u^\mu = dx^\mu/d\lambda$. Hence null particles remain null, but time-like lines have a variable u^2.

We see that matter no longer follows geodesics. However all bodies with the same set of initial conditions fall in the same way. A weak form of the equivalence principle is therefore satisfied. In particular there is no conflict between these theories and the Eotvos experiment. In [23] we shall investigate the impact of these effects upon the standard tests of gravitational light deflection, and the perihelium of Mercury. Here, however, we limit ourselves to integrating the geodesic equation in the local free-falling frame, or in flat space-time. Then (65) produces the Lagrangian

$$\mathcal{L} = e^{b\psi}(-(\dot{x}^0)^2 + \dot{x}^2) \qquad (67)$$

where dots represent $d/d\lambda$. There are three conserved quantities: $E = \dot{x}^0 e^{b\psi}$, $p = \dot{x} e^{b\psi}$, and $\mathcal{L} = -1$, from which we may conclude

$$\frac{v^2}{c^2} = \frac{p^2}{E^2} = 1 - \frac{e^{b\psi}}{E^2} \qquad (68)$$

As a result the particle's gamma factor

$$\gamma^2 = \frac{1}{1 - v^2/c^2} \propto c^{-b} \qquad (69)$$

If $b \neq 0$ the field ψ will therefore accelerate or brake particles.

VIII. FIXED SPEED OF LIGHT DUALS

We now identify which of our theories are simply well-known fixed c theories subject to a change of units. By doing so we will also expose the undesirable complication of the fixed c picture in all other cases.

Let us first rewrite our theories in units in which c, $\hbar$, and G are fixed, but the couplings g are variable, thereby mapping VSL theories into "Bekenstein" changing charge theories. Recalling that in VSL units we have $\alpha_i \propto g_i \propto \hbar c \propto c^q$ (cf. Eqn. (41)), we should perform the following change of units:

$$d\hat{t} = dt e^{(3-\frac{q}{2})\psi} \qquad (70)$$
$$d\hat{x} = dx e^{(2-\frac{q}{2})\psi} \qquad (71)$$
$$d\hat{E} = dE e^{(-2-\frac{q}{2})\psi} \qquad (72)$$

In the new units $\hat{c}$, $\hat{\hbar}$, and $\hat{G}$ are constant, $\hat{g} \propto e^{\frac{q}{2}\psi}$, and indeed $\hat{\alpha} = \alpha \propto c^q$. Subjecting a VSL minimally coupled matter action to this transformation leads to an action

very far from minimal coupling. Indeed all matter fields, eg ϕ, transform like

$$\hat{\phi} = \phi e^{-2\psi} \qquad (73)$$

Hence all kinetic terms become rather contorted, since in the new units

$$\partial_\mu \phi \rightarrow \partial_{\hat{\mu}} \hat{\phi} + 2\hat{\phi}\partial_{\hat{\mu}}\psi \qquad (74)$$

This leads to complex additions to mass and interaction terms. For gauged fields we have

$$D_\mu \phi \rightarrow D_{\hat{\mu}} \hat{\phi} + 2\hat{\phi}\partial_{\hat{\mu}}\psi \qquad (75)$$

leading to similar complications, and to breaking of standard gauge invariance. In conclusion we can transform a VSL theory which is minimally coupled to matter (up to the $b \neq 0$ factor) into a fixed c, $\hbar$ and G theory. However the result is a rather unnatural construction, quite distinct from the changing charge theories previously discussed. None of our theories is a standard changing charge theory in disguise; indeed choosing a standard changing g_i picture for them is undesirable.

The above may be avoided if we map our VSL theories into theories in which c and $\hbar$ are constants, but G may vary. Then in order to preserve minimal coupling all matter fields should remain unaffected by the unit transformation, eg $\hat{\phi} = \phi$ This requires

$$d\hat{t} = dt e^{(1-\frac{q}{2})\psi} \qquad (76)$$
$$d\hat{x} = dx e^{-\frac{q}{2}\psi} \qquad (77)$$
$$d\hat{E} = dE e^{-\frac{q}{2}\psi} \qquad (78)$$

and so we have that

$$\frac{\hat{G}}{G} = e^{-4\psi} \qquad (79)$$

While this ensures minimal coupling for all quantum fields, it does not do the job for classical point particles, if $q \neq 0$. With the above change of units one should make the identification

$$\hat{\phi}_{bd} = \frac{1}{\hat{G}} = e^{4\psi} \qquad (80)$$

and write down the transformed action:

$$\hat{S} = \int d^4\hat{x} \sqrt{-\hat{g}} (\hat{\phi}_{bd}^{\frac{a+q}{4}} (\hat{R} - 2\hat{\Lambda})$$
$$- \frac{f(\hat{\phi}_{bd})}{\hat{\phi}_{bd}} \nabla_\mu \hat{\phi}_{bd} \nabla^\mu \hat{\phi}_{bd} + \frac{16\pi G_0}{c_0^4} \hat{\phi}_{bd}^{\frac{b+q}{4}} \hat{\mathcal{L}}_m) \qquad (81)$$

We see that only theories for which $b + q = 0$ are scalar-tensor theories in disguise. If $q = 0$ all structure "constants" α_i are constant, and indeed for $b = 0$ (and so $a = 4$) we can recognize in $\hat{S}$ the Brans-Dicke action. However we see that there are also changing α theories

which are really Brans Dicke theories in unusual units: theories with $b = -q \neq 0$. Such theories are dilaton theories. On the contrary, if $b + q \neq 0$ we have theories which can never be mapped into dilaton theories.

In addition one may perform conformal transformations upon VSL theories, mapping them into other VSL theories with different a and b. The relevant formulae shall be given in [23]. By means of conformal transformations it is always possible to write action (44) as a scalar-tensor theory, if the matter Lagrangian is homogeneous in the metric. The latter, however, is clearly not true, carrying with it the crucial implication that there is only one frame in which the coupling to matter is of the form $e^{b\psi}\mathcal{L}_m$, with $\mathcal{L}_m$ independent of ψ. Thus, the much heralded equivalence between conformal frames is broken as soon as matter is added to gravity and ψ (a point clearly made by [34]). One may recognize $a = 4$, $b = 0$ as the Jordan's frame, $a = 0$, $b = -4$ as the Einstein's frame, and $a = b = 1$ as the tree-level string frame. We should also note that the general class of couplings we have considered is contained within the theories proposed by Damour and Polyakov [35] as representing low-energy limits to string theory, beyond tree-level. More specifically, using the notation of [35], our theories are those for which $B_i(\Phi)$ is the same for all the matter fields.

In spite of these comments, in [23] we shall make use of conformal transformations to isolate the geodesic frame: the frame in which free-falling charge-free particles follow geodesics. This can always be defined because the Lagrangian of these particles is indeed homogeneous in the metric. While this trick simplifies some calculations, one should always bear in mind that the geodesic frame only looks simpler because a lot of gargabe is swept under the carpet by not writing the Lagrangian for all the other matter fields. Minimal coupling to all forms of matter always picks up a preferred frame, which is not the geodesic frame unless $b = 0$.

IX. EMPTY SPACE-TIME

The analogue of Minkowski space-time may be derived by setting $T^\mu_\nu = \Lambda = 0$ in Equation (52). We should also set $a = 0$ and $\kappa = 0$ so as to switch off the gravitational effects of ψ. Then $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, using an x^0 coordinate.

The speed of light can be found from (54), which in coordinates in which ψ is homogeneous becomes $\ddot{\psi} = 0$. This leads to $\dot{\psi} = \frac{1}{R}$, where R is an integration constant with dimensions of length (R can be positive or negative). If we only use coordinates in which ψ is homogeneous (or, as we shall see, if we stay close to the origin compared to the distance R) then a global time coordinate t may be defined. In terms of it we have:

$$\frac{1}{c^2}\frac{dc}{dt} = \frac{1}{R} \tag{82}$$

which integrates to

$$c = \frac{c_0}{1 + \frac{c_0 t}{R}} \tag{83}$$

This is nothing but c near the origin in Fock-Lorentz space-time, in which

$$c(\mathbf{r}, t; \mathbf{n}) = \frac{c_0}{1 + \frac{c_0 t}{R}}\left(\mathbf{n} + \frac{\mathbf{r}}{R}\right) \tag{84}$$

Even though global coordinates cannot be generally defined if c varies, we find that this case is special. Relations

$$d\hat{t} = \frac{dt}{\left(1 + \frac{c_0 t}{R}\right)^{N+1}}$$

$$d\hat{r} = \frac{d\mathbf{r}}{\left(1 + \frac{c_0 t}{R}\right)^N} \tag{85}$$

(corresponding to $\alpha = N+1$ and $\beta = N$) may be recovered from

$$\hat{t} = \frac{R}{Nc_0}\left(1 - \frac{1}{\left(1 + \frac{c_0 t}{R}\right)^N}\right) \tag{86}$$

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{\left(1 + \frac{c_0 t}{R}\right)^N} \tag{87}$$

(with $N > 0$) as long as $|\mathbf{r}| \ll |R|$. Hence, near the origin there are global varying c coordinates t and $\mathbf{r}$. Global transformation laws between inertial frames may be derived for these coordinates by writing global Lorentz transformations for $\hat{t}$ and $\hat{\mathbf{r}}$ and then re-expressing them in terms of t and $\mathbf{r}$.

The case $N = 1$ is particularly simple. It corresponds to $q = 2$ for a fixed G representation ($\alpha \propto c^2$), or $q = -2$ in a minimal varying G representation ($\alpha \propto 1/c^2$). In these cases we have global Lorentz transformations for coordinates

$$\hat{t} = \frac{t}{1 + \frac{c_0 t}{R}} \tag{88}$$

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{1 + \frac{c_0 t}{R}} \tag{89}$$

These can be inverted into transformations for t and $\mathbf{r}$:

$$t' = \frac{\gamma\left(t - \frac{\mathbf{v}\cdot\mathbf{r}}{c_0^2}\right)}{1 - (\gamma - 1)\frac{c_0 t}{R} + \gamma\frac{\mathbf{v}\cdot\mathbf{r}}{Rc_0}} \tag{90}$$

$$\mathbf{r}'_{||} = \frac{\gamma\left(\mathbf{r}_{||} - \frac{\mathbf{v}}{c_0}t\right)}{1 - (\gamma - 1)\frac{c_0 t}{R} + \gamma\frac{\mathbf{v}\cdot\mathbf{r}}{Rc_0}} \tag{91}$$

$$\mathbf{r}'_\perp = \frac{\mathbf{r}_\perp}{1 - (\gamma - 1)\frac{c_0 t}{R} + \gamma\frac{\mathbf{v}\cdot\mathbf{r}}{Rc_0}} \tag{92}$$

where v is the velocity between two inertial frames at the origin at $t = 0$ (the velocity between two inertial frames varies in space and time and is proportional to c [20,21]). The transformation we have just obtained is the Fock-Lorentz transformation.

10

This is an interesting result! The Fock-Lorentz transformation was first derived by Vladimir Fock in his textbook [19] as a pedagogic curiosity. Special relativity may be derived from two postulates: the principle of relativity and the principle of constancy of the speed of light. The latter may be replaced by the requirement that the transformation be linear. Fock examined the effects of dropping the second postulate while keeping the first. He thus arrived at a fractional linear transformation identical with the one we have just derived.

We have just produced an alternative derivation, based on our dynamical equations for the field ψ. The constant R in the Fock Lorentz transformation appears as an integration constant in our solution. Some features of the Fock transformation, not accommodated by our theory (such as an anisotropic c), can be neglected if we stay close to the origin. Similarly some features of our theory not present in Fock's theory (such as non-integrability of infinitesimals) can be ignored in the same region. Hence it is not surprising that we have arrived at the same construction.

The Fock-Lorentz transformation has a number of interesting properties, and one of them is crucial for understanding VSL theories. If we consider a proper time interval Δt_0 (referred to the origin) we find that this is seen in the lab frame as

$$\Delta t = \frac{\Delta t_0}{(1 + c_0\Delta t_0/R)\gamma - c_0\Delta t_0/R} \tag{93}$$

which is qualitatively very different from the usual twin paradox expression. In the standard theory the only invariant non-zero time lapse is infinity. In Fock's theory such a role is played by $\Delta t_0 = -R/c_0$; in contrast infinity is no longer invariant but can mapped into finite times and vice-versa. Suitable particle life-times may be mapped to infinity (ie: stability) by a Fock-Lorentz transformation.

Closer inspection shows that if we look at these theories from a fixed c perspective $t = -R/c_0$ is indeed mapped into $\hat{t} = \infty$ for $R < 0$ (or $\hat{t} = -\infty$ for $R > 0$). This is obvious from (85) but also true for other values of N. Given that the two representations are globally very different one must ask which representation is more physical.

A. Interaction clocks and trans-eternal times

Clearly a change of units transforms our construction into plain Minkowski space-time. Then why not use the fixed c representation? The point is that the correspondence is only local. We can extend the VSL empty solution beyond $t = t_{max} = -R/c_0$, for $R < 0$. Such extension corresponds to extending Minkowski space-time beyond $t = \infty$. The choice between the two representations is therefore dependent on whether this extension is physical or not.

Let us first examine $t \to t_{max}$ in units in which c varies. In this picture c goes to infinity at t_{max}; but this has implications on the time-scales of processes mediated by all interactions. Decay times, rates of change, etc, all depend on the α_i. A typical time scale associated with a given interaction with energy Q is

$$\tau = \frac{\hbar}{\alpha^2 Q} \tag{94}$$

In a minimal VSL theory $Q \propto \hbar c \propto c^q$, $\alpha \propto c^q$, and so $\tau \propto 1/c^{2q+1}$. But our sensation of time flow derives precisely from change, and this is imparted by interactions and their rates. One may therefore argue that a more solid definition of time should be tied to the rates τ, and that a more physical clock should be obtained by making it tick to τ. Like all other definitions of time, this definition should not affect physics (which is dimensionless); however it may lead to a clearer picture.

In the varying c picture, the number of cycles of an interaction clock as $t \to t_{max}$ is

$$\int^{t_{max}} \frac{dt}{\tau} \tag{95}$$

which converges if $q < 0$, that is if all α_i go to zero (all interactions switch off). Hence our claim that the space is extendable beyond $t = t_{max}$ is physically meaningful, if $q < 0$.

Let us now examine the same situation in fixed c units. Even though c and $\hbar$ are now fixed, this is not really just Minkowski space-time. At the very least all charges g_i must now be variable, to produce the same changing α_i. If we want to keep all parameters in (94) constant except for the α_i we should change units in the following way:

$$d\hat{t} = dt e^{(1-2q)\psi} \tag{96}$$
$$d\hat{x} = dx e^{-2q\psi} \tag{97}$$
$$d\hat{E} = dE e^{-q\psi} \tag{98}$$

For $q < 0$ we have that t_{max} is indeed mapped into $\hat{t} = \infty$. However we find that the number of ticks of an interaction clock as we approach $\hat{t} = \infty$

$$\int^{\infty} \frac{d\hat{t}}{\hat{\tau}} \tag{99}$$

converges. Hence the temporal infinity of "Minkowski" space-time in this theory is spurious. Any natural process would slow down as "fixed-c time" went on. More and more of this "time" would be required for any interaction process to take place. Given that our sensation of time flowing is attached to these processes, we could claim that conversely we would feel that "fixed-c" time would start to go faster and faster. The fact that a finite number of physical ticks is required to reach $\hat{t} = \infty$ means that any observer could in fact flow through eternity. Such Minkowski space-time is physically extendable beyond $t = \infty$.

11

We have found the first example of a situation in which the fixed c representation, while locally equivalent to a varying c representation, may be globally misleading. The advantage of varying c units in this case is that they locate at a finite time distance what can in fact be reached within a finite number of cycles of an interaction clock.

X. FAST-TRACKS IN VSL FLAT-SPACE

More fascinating still is the existence of high-c lines, which we shall call fast-tracks. These are flat space-time solutions, in theories in which ψ is driven by a potential. We first establish the possibility of such solutions. Let ψ be a complex scalar field, with a $U(1)$ symmetry which may or may not be gauged (we assume it's gauged in what follows). Let the speed of light be given by $c = c_0 e^{-|\psi|^2}$. With these modifications we also have to modify the terms in a and b in (44), but not if $a = b = 0$, as we shall assume. Let us also assume that the field is driven by a potential

$$\mathcal{L}_\psi = -(D_\mu \psi)^\star (D^\mu \psi) - V(\psi) \tag{100}$$

$$V(\psi) = \frac{1}{\lambda_\psi^2} (|\psi|^2 - \psi_0^2)^2 \tag{101}$$

where ψ_0 is the field's vacuum expectation value, and λ_ψ is the Compton wave-length of ψ.

Let us consider a Nielsen-Olesen vortex solution to this theory, that is a solution with a boundary condition:

$$\psi = \psi_0 e^{in\theta} \qquad \text{as} \qquad r \to \infty \tag{102}$$

Such a solution is topologically stable. In the vortex's core, $|\psi| \approx 0$ and so the speed of light is c_0. The speed of light outside the core (which is $c_0 e^{-\psi_0^2}$) is therefore much smaller. The field ψ undergoes spontaneous symmetry breaking and the unbroken phase, realized in the string's core, displays a much larger speed of light. An approximate solution for $r \to \infty$ is

$$\psi = (\psi_0 + e^{-r/\lambda_\psi}) e^{in\theta} \tag{103}$$

Hence the string core has a width of order λ_ψ, which could easily be macroscopic; outside the core variations in the speed of light die off rapidly. The jump in the speed of light is exponential and depends only on ψ_0. For $\psi_0 \approx 3$, say, the speed of light could be ten orders of magnitude faster inside the string's core. The size of the core, and the jump in c, are related to independent parameters.

What would happen if an observer travelled along the string, inside its core? Let a cylinder of high-c connect two distant galaxies. Then inside the tube $v \propto c$ (cf. (68) with $b = 0$). Let us assume that $v \ll c$ so that no relativistic effects are present. Then the observer could move very fast between these two galaxies, returning without any time dilation effects having taken place. There would not be a twin paradox - clearly this situation, if realizable, is just what intergalactic travel is begging for. In practice, to avoid different aging rates between sedentary and the nomadic twins we should keep the aging pace τ fixed, ie: $q = -1/2$. Furthermore in order for the x^0 coordinate to track proper-time for all observers we should have $\alpha = 0$ (this point will be developed further in [23] in connection with radar echo delay experiments).

In a dual representation, in fixed c units, fast tracks are wormholes. If τ is to remain unchanged, and if c is to be fixed in the new units, then the distance between the galaxies must shrink by a corresponding factor (recall that in (1) $\alpha = 0$, and $\beta \neq 0$). Hence the fixed c dual of the VSL theories we have proposed contain wormhole like solutions even without the presence of gravitating matter. This is due to the fact that the gravitational action is indeed very complicated in the dual picture (notice that the required unit transformation is a combination of VSL and conformal transformations).

Elsewhere [36] we shall show how fast-tracks may appear in other theories, eg in the Bekenstein changing α theory. In such theories α is much smaller inside the string core, but all other couplings remain unchanged. Hence a nomadic twin will age much slower during the trip, since we age electromagnetically [37]. Strong interactions just provide the nuclei for all the atoms we are made of. But we are essentially made of stable nuclei. Hence if all our nuclei aged a million years we would not notice it. Naturally in such theories one cannot avoid different aging rates between nomadic and sedentary twins - the curse of space travel.

XI. BLACK HOLES WITH AN EDGE

In [23] we shall examine vacuum spherically symmetric solutions to all these theories. They have a common feature which can be illustrated by the well-known solution in Brans-Dicke theory (which is $a = 4$, $b = 0$). Using the isotropic form of the metric:

$$ds^2 = -f dx^{0^2} + g[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{104}$$

we have

$$f = f_0 \left(\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}} \right)^{2/\lambda} \tag{105}$$

$$g = g_0 (1 + B/r)^4 \left(\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}} \right)^{2(\lambda - C - 1)/\lambda} \tag{106}$$

$$\psi = \frac{-C}{4\lambda} \log \left(\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}} \right) \tag{107}$$

where f_0, g_0, B, and C, are constants, with:

$$\lambda = [(C + 1)^2 - C(1 - \omega_{bd} C/2)]^{1/2} \tag{108}$$

Expressions for these constants in terms of the black hole mass m and coupling ω_{bd} may be found in [32]. As we approach the horizon ($r_h = B$) we find that c goes to either zero or infinity (like $(r - r_h)^N$ with N related to ω_{bd}). The implication is obvious: for some parameters of the theory (in this case requiring $q \neq 0$[1]) no observer may enter the horizon. The number of cycles of an interaction clock trying to enter the horizon is given by:

$$\int^{r_h} \frac{dt}{\tau} = \int^{r_h} \frac{dr}{v\tau} = \int^{r_h} \frac{dr}{c^{2q+2}} \qquad (109)$$

which diverges for $2(q+1)N > 1$.

Again this phenomenon may be interpreted variously, depending on which units are used, but all interpretations lead to the same physical conclusion (which is dimensionless): particles are unable to enter the horizon. In VSL units particles cannot enter the horizon because they stop as c goes to zero. In fixed-c units they cannot enter the horizon because the time rates of all interactions go to zero (as all couplings go to infinite). Old age strikes before anything "has time" to enter the horizon.

Naturally finite sized bodies suffer from further effects, analogous to tidal stresses, since they will probe gradients in c. Since $v \propto c$ they get squashed if $c \to 0$, or get stretched otherwise. c-induced changes of pace also induce gradients of aging across finite-sized bodies.

A pedagogic illustration, studied further in [23], is a muon produced close to the black hole, moving towards its horizon. Such a set up is useful, for instance, when trying to convince skeptics of the physical validity of time dilation, or Lorentz contraction (eg. the fate of cosmic ray muons entering the atmosphere). To an Earth observer, if time dilation was not a physical effect the muon should never hit the surface of the Earth. From the point of view of the muon, if the atmosphere did not appear Lorentz contracted, it should have decayed before hitting the surface. The same set up will be of assistance here. No matter how close to the horizon the muon is produced, it never reaches it. In VSL units the muon stops as it tries to enter the horizon, because its speed is close to c, but c goes to zero. In fixed-c units the muon moves close to the (constant) speed of light, but its lifetime goes to zero as it tries to enter the horizon. From either perspective the muon can never enter the horizon.

"Horizon" is therefore a misnomer, and we relabel it an "edge": a boundary where c goes to zero sufficiently fast that no object may reach it. On physical grounds we should postulate that regions beyond the edge be excised from the manifold. Then VSL manifolds may have an edge.

We arrive at a similar conclusion to Section IX. VSL and fixed-c units are locally but not globally equivalent. The VSL picture may be globally more clear (in the case $a = 4$, only if $q \neq 0$). It builds into space-time the topology perceived by actual physical processes, in this case excising regions which are physically inaccessible.

The implications for the theory of singularities are quite impressive. Even though we have a singularity at $r = 0$, it is physically inaccessible. One may be able to prove that all singularities are subject to the same constraint. This situation was discussed in [38]. It looks as if a stronger version of the cosmic censorship principle might apply to these theories.

XII. CONCLUSION

One must sympathise with the view that VSL theories are rendered objectionable by their outright violation of Lorentz invariance. However, previous attempts to make the Albrecht-Magueijo model "geometrically honest" were no less ugly than the original; and were useless for cosmology. In this paper we proposed a geometrically honest VSL theory, corresponding to a theory in which all fine structure constants are promoted to dynamical variables. A changing charge interpretation in unnecessarily complicated - so we adopt units in which c changes, leading to a simple picture. This should not scandalise anyone.

All these theories are locally Lorentz invariant, and covariant in a sense incorporated by a generalized structure. We find that physics lives on a fibre bundle. Usually physics takes place on the tangent bundle. At each point in space-time there is a tangent space, corresponding to free falling frames in which physics is Minkowskian. We have a similar construction, but in the new units the space is not the tangent space of any coordinate patch in the manifold. It is still a vector space - but it is not the tangent vector space, except in the rare cases where the change of units is holonomic [2].

Given that a change of units maps these structures to standard covariant and local Lorentz invariant theories, one may wonder why it is worth bothering. To answer this question, throughout this paper we examined these "dual" theories. For them c is a constant (as well as $\hbar$ and possibly G), but naturally other quantities must vary. Indeed all couplings must change, at fixed ratios. We therefore have a theory not dissimilar from Bekenstein's changing e theory, but such a picture is horribly

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[1]In plain Brans-Dicke ($a = 4, b = 0, q = 0$) we have that $c \to 0$ but $\tau \to \infty$ in such a way that particles may enter the horizon. Hence the discussion presented here does not apply, as one would expect.

[2] The situation is more complicated for a gravitation theory based upon Fock-Lorentz space-time. Now the physics' space at a given point is no longer a vector space, but a projective space. The fibre bundle multiplies the base manifold by a projective space at each point.

misleading for the following reasons.

Firstly all charges, not only e, will vary. But they vary at constant ratios, so that all changes may be attributed to a change in c alone. Hence the dual theory is a theory which promotes coupling constants to dynamical variables, but then only allows rather contrived variations, ie variations which may be absorbed into a changing c. It seems therefore more natural to consider a changing c description, even though the two descriptions are indeed operationally equivalent.

Secondly the minimal dynamics in the two frames is totally different. This results from the fact that the action has units, and therefore changes under a change of units mapping dual theories. The minimal Bekenstein-type of theories does not have the same coupling to gravity as appears in the minimal VSL formulation. Rewriting the Lagrangian of minimal VSL theories in fixed c units leads to an unpleasant mess (Section VIII).

Thirdly and more importantly, the correspondence between VSL and its duals is only local. Globally the VSL picture can be more clear. We gave two striking examples. Fock-Lorentz space-time is just a change of units applied to Minkowski space-time; however it contains $t > \infty$ extensions to Minkowski space-time which are physically accessible. The horizon of a black hole may be physically impenetrable, since c goes to zero. Calling it an edge, and excising the bit beyond the edge seems reasonable. In the dual picture no warning about the fact that a piece of the manifold is inaccessible is given. It is an afterthought to notice that all interaction strengths force the pace of aging to become very fast; thereby, for all practical purposes, preventing anything from entering the horizon.

Hence the VSL theories we have proposed are changing c theories simply because choosing units in which c varies leads to a simpler description. Their underlying geometrical structure is that of standard fixed c theories subject to a change of units; a fact undeniably placing them at the pinnacle of geometrical honesty. It remains to show that these theories, applied to cosmology, perform as well as the Albrecht and Magueijo model. Such is the purpose of [22]. In any case it is not difficult to guess the overall cosmological picture to emerge in these theories. We see that the presence of a cosmological constant Λ generally drives changes in c, which in turn convert the vacuum energy into radiation leading to a conventional Big Bang. However, such a Big Bang is free from the standard cosmological problems, including the cosmological constant problem. The fact that particle production occurs naturally in these theories ensures that we also solve the entropy problem.

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APPENDIX - BIMETRIC REALIZATION OF THE ALBRECHT-MAGUEIJO MODEL

A theory which emulates many of the features of the Albrecht and Magueijo model (except for breaking Lorentz invariance) is the following. Let there be two metrics, g coupling to gravitation and matter, and h coupling to the field c only. Then we may take the following action:

$$S = S_1 + S_2$$
$$S_1 = \int d^4x \sqrt{-g}\left(R - 2\Lambda + \frac{16\pi G}{c_0^4 e^{4\psi}}\mathcal{L}_m\right)$$
$$S_2 = \int d^4x \sqrt{-h}(H - 2\Lambda_h - \kappa h^{\mu\nu}\partial_\mu\psi\partial_\nu\psi) \qquad (110)$$

where $g_{\mu\nu}$ and $h_{\mu\nu}$ lead to two Einstein tensors $G_{\mu\nu}$ and $H_{\mu\nu}$, and Λ and Λ_h are their respective (geometrical) cosmological cosntants. Varying with respect to g, ψ, and h leads to equations of motion:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c_0^4 e^{4\psi}}T_{\mu\nu} \qquad (111)$$

$$\Box_h \psi = \frac{32\pi G}{c_0^4 e^{4\psi}\kappa}\sqrt{\frac{g}{h}}\mathcal{L}_m \qquad (112)$$

$$H_{\mu\nu} + \Lambda_h h_{\mu\nu} = -\frac{\kappa}{2}\left(\nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}h_{\mu\nu}\nabla_\alpha\psi\nabla^\alpha\psi\right) \quad (113)$$

Hence we may derive from an action principle the property that the field ψ does not contribute to the stress-energy tensor which acts as a source to normal space-time curvature. In [22,23] we shall highlight some curiosities pertaining to these theories.